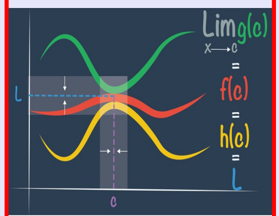
Calculus I

Lecture 6



Feb 19-8:47 AM

Class Quiz 6

Box Your Final Ans.

Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for $f(x) = ax^2 + bx$.

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) - ax^2 - bx}{h}$$

$$= \lim_{h \to 0} \frac{a(x^2 + 2xh + h^2) + b(x+h) - ax^2 - bx}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{ax^2} + 2axh + ah^2 \cancel{+bx} + bh \cancel{-ax^2} \cancel{-bx}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2ax + ah + b)}{\cancel{h}} = \lim_{h \to 0} (2ax + ah + b)$$

$$= 2ax + a(0) + b$$

$$= \boxed{2ax + b}$$

Mar 3-7:46 AM

Evaluate

$$\lim_{x\to 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0} \quad \text{I.F.}$$

$LCD = 2(x+2)$

$$= \lim_{x\to 0} \frac{2 - (x+2)}{x \cdot 2(x+2)} = \lim_{x\to 0} \frac{\cancel{2} - \cancel{x}^{-1} - \cancel{2}}{2\cancel{x}(x+2)}$$

$$= \lim_{x\to 0} \frac{-1}{2(x+2)} = \boxed{\frac{-1}{4}}$$

Mar 3-9:04 AM

$f(x) = \frac{x^2-9}{x^3-27}$ if $x \neq 3$

$f(x) = K$ if $x = 3$

$$\Rightarrow f(x) = \begin{cases} \frac{x^2-9}{x^3-27} & \text{if } x \neq 3 \\ K & \text{if } x = 3 \end{cases}$$

1) find $f(3) = \boxed{K}$

2) find $\lim_{x\to 3} f(x)$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x^3-27} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x^2+3x+9)}$$

$$= \lim_{x\to 3} \frac{x+3}{x^2+3x+9}$$

$$= \frac{3+3}{3^2+3(3)+9} = \frac{6}{27} = \boxed{\frac{2}{9}}$$

For $f(x)$ to be continuous at $x = 3$

$$\lim_{x\to 3} f(x) = f(3)$$

$$\boxed{\frac{2}{9} = K}$$

Mar 3-9:10 AM

Suppose $\sin(x-1) \le f(x) \le \cos\left(\frac{\pi}{2}x\right)$

Find $\lim_{x \to 1} f(x)$

$$\lim_{x \to 1} \sin(x-1) = \sin(1-1) = \sin 0 = 0$$

$$\lim_{x \to 1} \cos\frac{\pi}{2}x = \cos\frac{\pi}{2}(1) = \cos\frac{\pi}{2} = 0$$

by S.T. $\lim_{x \to 1} f(x) = \boxed{0}$

Mar 3-9:19 AM

Evaluate

1) $\lim_{x \to 0} \dfrac{\sin 3x}{\sin 8x} = \dfrac{\sin 0}{\sin 0} = \dfrac{0}{0}$ I.F.

$$= \lim_{x \to 0} \frac{3\frac{\sin 3x}{3x}}{8\frac{\sin 8x}{8x}} = \frac{3}{8} \cdot \frac{\lim_{x \to 0} \frac{\sin 3x}{3x} \to 1}{\lim_{x \to 0} \frac{\sin 8x}{8x} \to 1} = \boxed{\frac{3}{8}}$$

$$\boxed{\begin{array}{l} \lim_{h \to 0} \dfrac{\sin h}{h} = 1 \\[2em] \lim_{h \to 0} \dfrac{1 - \cos h}{h} = 0 \end{array}}$$

2) $\lim_{x \to \pi/4} \dfrac{1 - \cos\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{h \to 0} \dfrac{1 - \cos h}{h} = \boxed{0}$

Let $h = x - \frac{\pi}{4}$

$x \to \frac{\pi}{4}$, $h \to 0$

Mar 3-9:25 AM

Find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $f(x)=\sin x$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x - \sin x}{0} = \frac{0}{0} \text{ I.F.}$$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cosh - 1) + \cos x \sinh}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x(\cosh - 1)}{h} + \frac{\cos x \sinh}{h}\right]$$

$$= \lim_{h\to 0} \frac{\sin x [\cosh - 1]}{h} + \lim_{h\to 0} \frac{\cos x \sinh}{h}$$

$$= \sin x \boxed{\lim_{h\to 0} \frac{\cosh - 1}{h}} + \cos x \boxed{\lim_{h\to 0} \frac{\sinh}{h}}$$

$$\boxed{\lim_{h\to 0} \frac{\cosh - 1}{h}} \to 0 \qquad \boxed{\lim_{h\to 0} \frac{\sinh}{h}} \to 1$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \boxed{\cos x}$$

Mar 3-9:33 AM

For $\varepsilon > 0$, find a $\delta > 0$ such that $\lim_{x\to 1}(2x+6) = 8$.

$$\lim_{x\to 1}(2x+6) = 8$$

$f(x) = 2x+6$ $\quad L = 8$ ✓ $\quad a = 1$

$$\lim_{x\to a} f(x) = \lim_{x\to 1}(2x+6) = 2(1)+6 = 8 \checkmark$$

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$|2x+6-8| < \varepsilon$ " $|x-1| < \delta$

$|2x-2| < \varepsilon$

$|2(x-1)| < \varepsilon$

$2|x-1| < \varepsilon$

$|x-1| < \frac{\varepsilon}{2}$ $\quad \to \delta = \frac{\varepsilon}{2}$

$\varepsilon = 1 \to \delta = \frac{1}{2}$

$\varepsilon = \frac{1}{2} \to \delta = \frac{1}{4}$

$\varepsilon = 2 \to \delta = 1$

Mar 3-9:40 AM

For $\varepsilon > 0$, find a $\delta > 0$ such that $\lim_{x \to -2}\left(\frac{-1}{2}x - 1\right) = 0$.

$$\lim_{x \to -2}\left(\frac{-1}{2}x - 1\right) = 0$$

$f(x) = \frac{-1}{2}x - 1$ $\quad L = 0 \checkmark$ $\quad a = -2$

$$\lim_{x \to -2}\left(\frac{-1}{2}x - 1\right) = \frac{-1}{2}(-2) - 1 = 1 - 1 = 0 \checkmark$$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{-1}{2}x - 1 - 0\right| < \varepsilon$ " $|x - (-2)| < \delta$

$\left|\frac{-1}{2}x - \frac{2}{2}\right| < \varepsilon$ " $|x + 2| < \delta$

$\left|\frac{-1}{2}(x + 2)\right| < \varepsilon$ " $|x + 2| < \delta$

$\left|\frac{-1}{2}\right| |x + 2| < \varepsilon$ $\quad \boxed{\delta = 2\varepsilon}$

$\frac{1}{2}|x + 2| < \varepsilon$

Multiply by 2

$|x + 2| < 2\varepsilon$

Mar 3-9:47 AM

Prove $\lim_{x \to 2}(x^2 + 3x) = 10$

1) $f(x) = x^2 + 3x$ $\quad L = 10 \checkmark$ $\quad a = 2$

2) Verify the limit.

$$\lim_{x \to 2}(x^2 + 3x) = 2^2 + 3(2) = 4 + 6 = 10 \checkmark$$

3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 + 3x - 10| < \varepsilon$ " $|x - 2| < \delta$

$|(x+5)(x-2)| < \varepsilon$ " $|x - 2| < \delta$

$\underbrace{|x+5|}_{\text{Bound}}\ \underbrace{|x-2|}_{\text{Keep}} < \varepsilon$ " $|x - 2| < \delta$

If $|x+5| < C$, then $|x+5||x-2| < C|x-2| < \varepsilon$

$|x - 2| < \frac{\varepsilon}{C}$

If $\delta \leq 1$, then $|x - 2| < 1$

$-1 < x - 2 < 1$

Add 7

$-1 + 7 < x - 2 + 7 < 1 + 7$

$6 < x + 5 < 8 \Rightarrow |x + 5| < 8$

$\boxed{\delta = \min\left\{1, \frac{\varepsilon}{8}\right\}}$

$\varepsilon = 1 \to \delta = \min\left\{1, \frac{1}{8}\right\} = \frac{1}{8}$

$\varepsilon = 12 \to \delta = \min\left\{1, \frac{12}{8}\right\} = 1$

$\varepsilon = 2 \to \delta = \min\left\{1, \frac{2}{8}\right\} = \frac{2}{8} = \frac{1}{4}$

Mar 3-9:56 AM

Prove $\lim_{x \to 4} x^2 = 16.$

1) $f(x) = x^2$ $L = 16 \checkmark$ $a = 4$

$\lim_{x \to 4} x^2 = 4^2 = 16 \checkmark$

2) Verify the limit

3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 - 16| < \varepsilon$ 〃 $|x - 4| < \delta$

$|(x+4)(x-4)| < \varepsilon$ 〃 $|x - 4| < \delta$

$\underbrace{|x+4|}_{\text{Bound}} \underbrace{|x-4|}_{\text{Keep}} < \varepsilon$

If $|x+4| < C$, then $|x+4||x-4| < C|x-4| < \varepsilon$

$|x-4| < \frac{\varepsilon}{C}$

For polynomial function,

we want $\delta \le 1$

$|x-4| < 1$

$-1 < x - 4 < 1$

Add 8

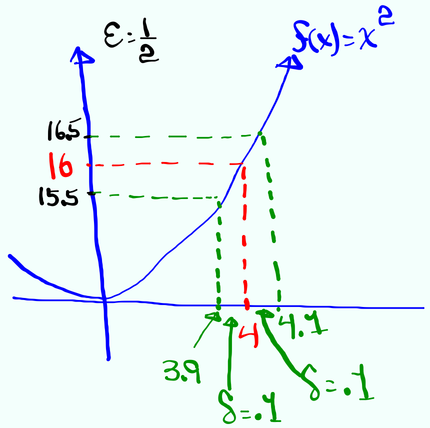
$-1 + 8 < x - 4 + 8 < 1 + 8$

$7 < x + 4 < 9 \Rightarrow |x+4| < 9$

C

$\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$

Mar 3-10:20 AM



$x^2 = 15.5$

$x = \sqrt{15.5} \approx 3.9$

$x^2 = 16.5$

$x = \sqrt{16.5} \approx 4.1$

$\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$

$= \min\left\{1, \frac{\frac{1}{2}}{9}\right\}$

$= \min\left\{1, \frac{1}{18}\right\}$

$= \frac{1}{18} = .0\overline{5}$

$\approx \boxed{.1}$

Mar 3-10:29 AM

Prove $\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$

1) $f(x) = \frac{1}{x}$ $\quad L = 2$ ✓ $\quad a = \frac{1}{2}$

2) verify the limit

$$\lim_{x \to \frac{1}{2}} \frac{1}{x} = \frac{1}{\frac{1}{2}} = 1 \div \frac{1}{2} = 1 \cdot \frac{2}{1} = 2 ✓$$

3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{1}{x} - 2\right| < \varepsilon$ 〃 $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{1-2x}{x}\right| < \varepsilon$ 〃 $\left|x - \frac{1}{2}\right| < \delta$

$|a-b| = |b-a|$

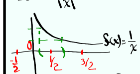
$\left|\frac{2x-1}{x}\right| < \varepsilon$

$\left|\frac{2(x-\frac{1}{2})}{x}\right| < \varepsilon$ 〃 $\left|x - \frac{1}{2}\right| < \delta$

$$\underbrace{\frac{2}{|x|}}_{\text{Bound}} \underbrace{\left|x - \frac{1}{2}\right|}_{\text{Keep}} < \varepsilon$$

If $\frac{2}{|x|} < C$, then $\frac{2}{|x|}\left|x - \frac{1}{2}\right| < C\left|x - \frac{1}{2}\right| < \varepsilon$

$\left|x - \frac{1}{2}\right| < \frac{\varepsilon}{C}$



$\delta < \frac{1}{2}$

$\left|x - \frac{1}{2}\right| < \frac{1}{4}$

Pick $\boxed{\delta \le \frac{1}{4}}$

$-\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}$

Add $\frac{1}{2}$

$-\frac{1}{4} + \frac{1}{2} < x - \frac{1}{2} + \frac{1}{2} < \frac{1}{4} + \frac{1}{2}$

$\frac{1}{4} < x < \frac{3}{4}$

$4 > \frac{1}{x} > \frac{4}{3}$

$\frac{4}{3} < \frac{1}{x} < 4$

multiply by 2

$\frac{8}{3} < \frac{2}{x} < 8$

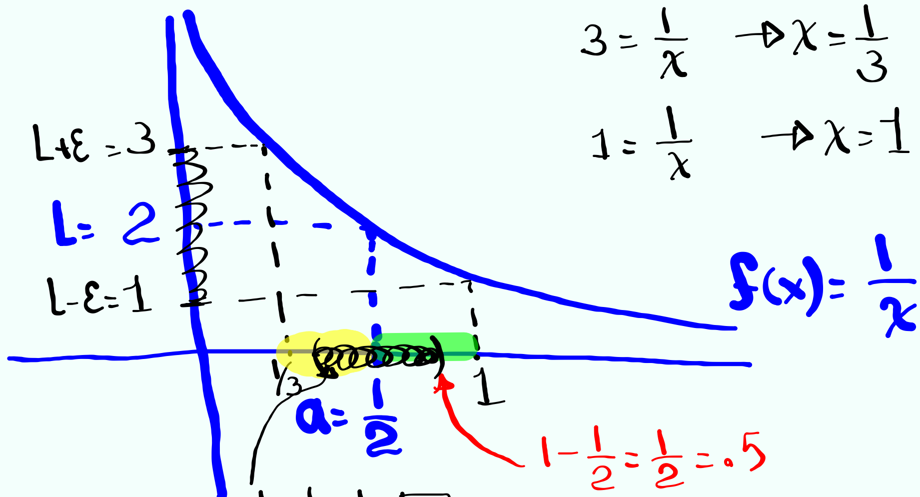
$\frac{2}{|x|} < 8$

C

$\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{8}\right\}$

$\boxed{\varepsilon = 1} \to \delta = \min\left\{\frac{1}{4}, \frac{1}{8}\right\} = \frac{1}{8}$

Mar 3-10:34 AM



$3 = \frac{1}{x} \to x = \frac{1}{3}$

$1 = \frac{1}{x} \to x = 1$

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \boxed{.17}$

$1 - \frac{1}{2} = \frac{1}{2} = .5$

Pick $\boxed{\delta = .17}$

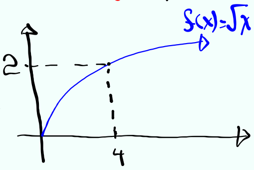
$\varepsilon = 1$

$\delta = \min\left\{\frac{1}{4}, \frac{1}{8}\right\} = \frac{1}{8} = \boxed{.125}$

Mar 3-10:53 AM

Prove $\lim_{x \to 4} \sqrt{x} = 2$

1) $f(x) = \sqrt{x}$ $\quad L = 2$ ✓ $\quad a = 4$



2) verify the limit

$\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$ ✓

3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|\sqrt{x} - 2| < \varepsilon$ " $|x-4| < \delta$

$\left| \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2} \right| < \varepsilon$ " $|x-4| < \delta$

$\left| \frac{x-4}{\sqrt{x}+2} \right| < \varepsilon$ " $|x-4| < \delta$

keep

$\boxed{\frac{1\,|x-4|}{|\sqrt{x}+2|}} < \varepsilon$

Bound $\frac{1}{|\sqrt{x}+2|}$

Mar 3-10:58 AM

Open notes class Quiz 7

Prove $\lim_{x \to -2} (5x - 3) = -13.$

$f(x) = 5x - 3$ $\quad L = -13$ ✓ $\quad a = -2$

$\lim_{x \to -2} (5x-3) = 5(-2) - 3 = \boxed{-13}$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|5x - 3 - (-13)| < \varepsilon$ " $|x - (-2)| < \delta$

$|5x - 3 + 13| < \varepsilon$ " $|x+2| < \delta$

$|5x + 10| < \varepsilon$ " $|x+2| < \delta$

$|5(x+2)| < \varepsilon$

$5|x+2| < \varepsilon \to |x+2| < \frac{\varepsilon}{5}$

Pick $\delta = \frac{\varepsilon}{5}$

Mar 3-11:06 AM